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Wigner Distribution Function for the Time-Dependent Quadratic-Hamiltonian Quantum System using the Lewis–Riesenfeld Invariant Operator

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We investigate the Wigner distribution function of the general time-dependent quadratic-Hamiltonian quantum system with the Lewis–Riesenfeld invariant operator method. The Wigner distribution function of the system in Fock state, coherent state, squeezed state, and thermal state are derived. We apply our study to the one-dimensional motion of a Brownian particle and to the driven oscillator with strongly pulsating mass.

KEY WORDS: Wigner distribution function; time-dependent Hamiltonian system; density operator; parity operator.
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1. INTRODUCTION

The study of quantum properties for the time-dependent quadratic Hamiltonian system (TDQHS) such as harmonic oscillator with time-variable mass and/or frequency has attracted considerable interest in the literature (Ji and Kim, 1996; Song, 2000; Yeon *et al.*, 1997; Nieto and Truax, 2001; Choi, 2004a,b, 2003) after the invention of the LR (Lewis–Riesenfeld) invariants (Lewis, 1967; Lewis and Riesenfeld, 1969) of the time-dependent harmonic oscillator. The LR invariants can be applicable to the derivation of the quantum solution for the TDQHS. In the previous paper, we derived exact wave function, energy eigenvalue, fluctuation of canonical variables, and uncertainty relation for the TDQHS (Choi, 2003, 2004a,b). We also investigated coherent state (Choi, 2004a), squeezed state (Choi, 2004b) and thermal state (Choi, 2003) of the TDQHS.

In this paper, we apply a phase space distribution function of quantum mechanics, the so-called Wigner distribution function proposed by Wigner (Wigner,

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1932) to the general TDQHS. Even though the Wigner distribution function offers a joint probability for position and momentum, it turned out that this function allows negative probability in some subset of phase space point (q, p). In fact, the only pure states for which the Wigner distribution function is everywhere positive are those for which the wave function satisfying Schrödinger equation is Gaussian (Hudson, 1974). For this reason, the Wigner distribution function is regarded as 'quasiprobability density.' In fact, it has been realized that quantum phase distribution function should be considered as just a mathematical tool that facilities quantum calculations since the concept of a joint probability at a phase space point (q, p) is not permissible due to the Heigenberg uncertainty principle (Lee, 1995). It may be worth to point out that Wigner distribution function has proved proportional to the expectation value of the parity operator (Royer, 1977). The Wigner distribution function can be widely used to the description of quantum states in a variety of branches in physics such as quantum optics (Schleich, 2001; Abe and Suzuki, 1992), solid-state physics (Janssen and Zwerger, 1995), and nonlinear physics (Lee, 1995) since it is useful for studying the passage from quantum to classical mechanics. Li gave a fairly rigorous group theoretical derivation of Wigner distribution function (Li, 1994). Polychromatic paraxial wavefields and their color images on a screen are studied using Wigner distribution function by Wolf (Wolf, 1996). Wigner distribution functions defined on coadjoint orbits of a class of semi-direct product groups are constructed (Krasowska and Ali, 2003). Alonso *et al.* proposed a form of the Wigner distribution functions for Hamiltonian systems in spaces of constant negative curvature such as hyperboloids (Alonso *et al.*, 2002) and constant positive curvature such as spheres (Alonso *et al.*, 2003).

In Section 2, we discuss the Fock state of the general TDQHS and derive Fock state Wigner distribution function. We evaluated Wigner distribution functions in coherent state, in squeezed state, and in thermal state in Section 3. We applied our study into the quantum system of one-dimensional motion of Brownian particle and the driven oscillator with strongly pulsating mass in Section 4. In the last section, we give a summary of our developments of the previous sections.

2. WIGNER DISTRIBUTION FUNCTION IN FOCK STATE

In ref. (Choi, 2004a), we derived exact quantum states of the general TDQHS. We briefly review them for the time beng. We represent the Hamiltonian of the most general TDQHS in the form

$$\hat{H}(\hat{q}, \hat{p}, t) = A(t)\hat{p}^2 + B(t)(\hat{q}\hat{p} + \hat{p}\hat{q}) + C(t)\hat{q}^2 + D(t)\hat{q} + E(t)\hat{p} + F(t), \tag{1}$$

where $A(t) - F(t)$ are time-variable functions that are differentiable with respect to time. We suppose that $A(t) \neq 0$. Functions $D(t)$ and $E(t)$ are related to the driving force of the system and $F(t)$ is related to the reference point of potential energy. Although $F(t)$ is zero or constant in most case, we let it a function of time in order to study TDQHS with generalized form of Hamiltonian. By appropriate choice of $A(t) - F(t)$, the system described by Eq. (1) may be applied to various concrete physical problem (see Section 4).

In order to facilitate the investigation of the quantum state for the TDQHS, it is convenient to introduce LR invariant operator that given by (Choi, 2004a)

$$\hat{I} = \frac{\Omega^2}{4\rho^2(t)}(\hat{q} - q_p(t))^2 + \left[\rho(t)(\hat{p} - p_p(t)) + \frac{1}{2A}(2B\rho(t) - \dot{\rho}(t))(\hat{q} - q_p(t))\right]^2, \tag{2}$$

where Ω is arbitrary real constant and $\rho(t)$ is some time-variable real solution of the following differential equation

$$\ddot{\rho}(t) - \frac{\dot{A}}{A}\dot{\rho}(t) + \left(2\frac{\dot{A}B}{A} - 4B^2 + 4AC - 2\dot{B}\right)\rho(t) - \Omega^2 A^2 \frac{1}{\rho^3(t)} = 0, \tag{3}$$

and $q_p(t)$ and $p_p(t)$ are particular solutions of the classical equation of motion in coordinate and momentum space, respectively. We introduce annihilation operator of the form (Choi, 2004a)

$$\hat{a} = \sqrt{\frac{1}{\hbar\Omega}}\left\{\left[\frac{\Omega}{2\rho} + i\frac{1}{2A}(2B\rho - \dot{\rho})\right](\hat{q} - q_p) + i\rho(\hat{p} - p_p)\right\}, \tag{4}$$

and its adjoint $\hat{a}^\dagger$, creation operator. We can easily check that $[\hat{a}, \hat{a}^\dagger] = 1$. In terms of $\hat{a}$ and $\hat{a}^\dagger$, Eq. (2) is simplified to

$$\hat{I} = \hbar\Omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{5}$$

The wave function of the system that satisfy the Schrödinger equation is given by (Choi, 2004a)

$$\psi_n(q, t) = \phi_n(q, t)\exp[i\epsilon_n(t)], \tag{6}$$

where $\phi_n(q, t)$ is the eigenstate of the invariant operator $\hat{I}$:

$$\phi_n(q, t) = \sqrt[4]{\frac{\Omega}{2\rho^2\hbar\pi}}\frac{1}{\sqrt{2^n n!}}H_n\left[\sqrt{\frac{\Omega}{2\rho^2\hbar}}(q - q_p)\right] \times \exp\left\{\frac{i}{\hbar}p_p q - \frac{1}{2\rho\hbar}\left[\frac{\Omega}{2}\frac{1}{\rho} + \frac{i}{2A}(2B\rho - \dot{\rho})\right](q - q_p)^2\right\}, \tag{7}$$

and $\epsilon_n(t)$ total phase of the wave function:

$$\epsilon_n(t) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{A(t')\Omega}{\rho^2(t')}\,dt' \\ -\frac{1}{\hbar}\int_0^t \left[\mathcal{L}_p(q_p(t'),\dot{q}_p(t'),t') - \frac{E^2(t')}{4A(t')} + F(t')\right]dt', \tag{8}$$

with

$$\mathcal{L}_p(q_p(t'),\dot{q}_p(t'),t') = \frac{1}{4A(t')}\dot{q}_p^2(t') - \frac{B(t')}{A(t')}q_p(t')\dot{q}_p(t') \\ -\left(C(t') - \frac{B^2(t')}{A(t')}\right)q_p^2(t'). \tag{9}$$

If we put $\rho(t) = \Omega^{1/2}\rho_0(t)$, all the Ω's in Eqs. (7) and (8) disappear. Thus, we need not worry about the magnitude of the constant Ω (Choi, 2004b).

When we define the density operator ϱ as

$$\hat{\varrho} = \sum_{n,m}\varrho_{nm}|\psi_n\rangle\langle\psi_m|, \tag{10}$$

the corresponding Wigner distribution function is represented as

$$W(q,p,t) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\langle q-y|\hat{\varrho}|q+y\rangle e^{2ipy/\hbar}\,dy \\ = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\hat{\varrho}(q-y,q+y,t)e^{2ipy/\hbar}\,dy. \tag{11}$$

For pure state $\hat{\varrho}(q,q',t)$ is given by

$$\hat{\varrho}(q,q',t) = \psi_n(q,t)\psi_n^*(q',t), \tag{12}$$

so that we can express Eq. (11) as

$$W_n(q,p,t) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\psi_n^*(q+y,t)\psi_n(q-y,t)e^{2ipy/\hbar}\,dy. \tag{13}$$

Substitution of Eq. (6) into the above equation gives

$$W_n(q,p,t) = \frac{1}{\pi\hbar}\sqrt{\frac{\Omega}{2\rho^2\hbar\pi}}\frac{1}{2^n n!}\exp\left[-\frac{\Omega}{2\rho^2\hbar}(q-q_p)^2\right] \\ \times\int_{-\infty}^{\infty}H_n\left[\sqrt{\frac{\Omega}{2\rho^2\hbar}}(q-q_p+y)\right]H_n\left[\sqrt{\frac{\Omega}{2\rho^2\hbar}}(q-q_p-y)\right] \\ \times\exp\left\{-\frac{\Omega}{2\rho^2\hbar}y^2 + \frac{i}{\hbar}\left[2(p-p_p) + \frac{2B\rho-\dot{\rho}}{\rho A}(q-q_p)\right]y\right\}dy. \tag{14}$$

Using the known integral formula (Gradshteyn and Ryzhik, 1980)

$$\int_{-\infty}^{\infty} dz e^{-z^2} H_n(z+\alpha+\beta) H_n(z+\alpha-\beta) = 2^n \pi^{1/2} n! L_n[2(\beta^2-\alpha^2)], \quad (15)$$

we arrive at

$$W_n(q,p,t) = \frac{(-1)^n}{\pi\hbar} e^{-2I(q,p,t)/(\hbar\Omega)} L_n\left(\frac{4}{\hbar\Omega} I(q,p,t)\right), \quad (16)$$

where L_n is Laguerre polynomial and $I(q, p, t)$ classical invariant quantity which can be obtained from Eq. (2) by replacing canonical operators $\hat{q}$ and $\hat{p}$ with classical variables q and p:

$$\begin{aligned} I(q,p,t) = {} & \frac{\Omega^2}{4\rho^2(t)}(q-q_p(t))^2 \\ & + \left[\rho(t)(p-p_p(t)) + \frac{1}{2A}(2B\rho(t)-\dot{\rho}(t))(q-q_p(t))\right]^2. \end{aligned} \quad (17)$$

In case of standard harmonic oscillator, $I(q, p, t)$ in Eq. (16) becomes classical Hamiltonian of the system. Although Eq. (16) is real, it is not always positive. Hence we cannot consider it as a probability distribution. However, when integrated over either of two variables q and p, it permits to be probability distribution for the other (Lee, 1995)

$$\int_{-\infty}^{\infty} W_n(q,p,t)\, dq = |\psi_n(p,t)|^2, \quad (18)$$

$$\int_{-\infty}^{\infty} W_n(q,p,t)\, dp = |\psi_n(q,t)|^2. \quad (19)$$

We define the displacement operator in the form

$$\begin{aligned} \hat{D}(\mathcal{A}) \equiv \hat{D}(q,p) &= \exp(\mathcal{A}\hat{a}^\dagger - \mathcal{A}^*\hat{a}) \\ &= \exp\left\{\frac{i}{\hbar}[(p-p_p)(\hat{q}-q_p) - (q-q_p)(\hat{p}-p_p)]\right\}, \end{aligned} \quad (20)$$

where

$$\mathcal{A} = \sqrt{\frac{1}{\hbar\Omega}}\left\{\left[\frac{\Omega}{2\rho} + i\frac{1}{2A}(2B\rho-\dot{\rho})\right](q-q_p) + i\rho(p-p_p)\right\}. \quad (21)$$

We denote real and imaginary part of $\mathcal{A}$ as $\mathcal{A}_{\mathrm{R}}$ and $\mathcal{A}_{\mathrm{I}}$:

$$\mathcal{A} = \mathcal{A}_{\mathrm{R}} + i\mathcal{A}_{\mathrm{I}}, \quad (22)$$

$$\mathcal{A}_{\mathrm{R}} = \frac{1}{2\rho}\sqrt{\frac{\Omega}{\hbar}}(q-q_p), \quad (23)$$

$$\mathcal{A}_{\mathrm{I}} = \sqrt{\frac{1}{\hbar\Omega}} \left[\frac{1}{2A}(2B\rho - \dot{\rho})(q - q_p) + \rho(p - p_p) \right]. \tag{24}$$

In later discussion, we also use phase representation of $\mathcal{A}$:

$$\mathcal{A} = |\mathcal{A}|e^{i\Theta}, \tag{25}$$

$$|\mathcal{A}| = \left(\mathcal{A}_{\mathrm{R}}^2 + \mathcal{A}_{\mathrm{I}}^2\right)^{1/2}, \tag{26}$$

$$\Theta = \tan^{-1} \frac{\mathcal{A}_{\mathrm{I}}}{\mathcal{A}_{\mathrm{R}}}. \tag{27}$$

In terms of $|\mathcal{A}|$, Eq. (17) can be rewritten as

$$I = \hbar\Omega|\mathcal{A}|^2. \tag{28}$$

Canivell and Seglar derived a simple expression of the parity operator of the form (Canivell and Seglar, 1978)

$$\hat{U}_0 = \exp(i\pi\hat{a}^\dagger\hat{a}) = \sum_n (-1)^n |\psi_n\rangle\langle\psi_n|. \tag{29}$$

Some of the important properties of the $\hat{U}_0$ are (Bishop and Vourdas, 1994; Chountasis and Vourdas, 1998; Chountasis *et al.*, 1999)

$$\hat{U}_0|q\rangle = |-q\rangle, \qquad \hat{U}_0|p\rangle = |-p,\rangle \tag{30}$$

$$\hat{U}_0\hat{q}\hat{U}_0^\dagger = -\hat{q}, \qquad \hat{U}_0\hat{p}\hat{U}_0^\dagger = -\hat{p}, \tag{31}$$

$$\hat{U}_0\hat{D}(q,p)\hat{U}_0^\dagger = \hat{D}(-q,-p), \tag{32}$$

$$\hat{U}_0\hat{f}(\hat{a},\hat{a}^\dagger)\hat{U}_0^\dagger = f(-\hat{a},-\hat{a}^\dagger), \tag{33}$$

where $\hat{f}(\hat{a},\hat{a}^\dagger)$ is an arbitrary operator which is the function of $\hat{a}$ and $\hat{a}^\dagger$. The displaced parity operator is represented as (Chountasis and Vourdas, 1998)

$$\begin{aligned} \hat{U}(\mathcal{A}) \equiv \hat{U}(q,p) &= \hat{D}(q,p)\hat{U}_0\hat{D}^\dagger(q,p) \\ &= \sum_{n=0}^{\infty} (-1)^n |\psi_n;\mathcal{A}\rangle\langle\psi_n;\mathcal{A}| \\ &= \exp[i\pi(\hat{a}^\dagger - \mathcal{A}^*)(\hat{a} - \mathcal{A})], \end{aligned} \tag{34}$$

where

$$|\psi_n;\mathcal{A}\rangle = \hat{D}(\mathcal{A})|\psi_n\rangle. \tag{35}$$

Note that $\hat{U}(q,p)$ satisfies $\hat{U}^2(q,p) = 1$ and is Hermitian operator $\hat{U}^\dagger(q,p) = \hat{U}(q,p)$ (Chountasis and Vourdas, 1998). The action of the $\hat{U}(q,p)$

on the position and momentum operator give

$$\hat{U}(q,p)\hat{q}\hat{U}^{\dagger}(q,p) = -\hat{q} + 2\hbar A_1, \tag{36}$$

$$\hat{U}(q,p)\hat{p}\hat{U}^{\dagger}(q,p) = -\hat{p} + 2\hbar A_2, \tag{37}$$

where

$$A_1 = \frac{2}{\sqrt{\hbar\Omega}}\rho\mathcal{A}_{\mathrm{R}}, \tag{38}$$

$$A_2 = \frac{2}{\sqrt{\hbar\Omega}}\left(\frac{\Omega}{2\rho}\mathcal{A}_{\mathrm{I}} - \frac{1}{2A}(2B\rho - \dot{\rho})\mathcal{A}_{\mathrm{R}}\right). \tag{39}$$

We can also show that the action of $\hat{U}(q,p)$ on the position and momentum eigenstates result

$$\hat{U}(q,p)|q\rangle = e^{iA_3}e^{-2iA_2(q-\hbar A_1)}|-q+2\hbar A_1\rangle, \tag{40}$$

$$\hat{U}(q,p)|p\rangle = e^{iA_3}e^{2iA_1(p-\hbar A_2)}|-p+2\hbar A_2\rangle, \tag{41}$$

where

$$A_3 = \frac{4}{\sqrt{\hbar\Omega}}\left[\left(\frac{\Omega}{2\rho}\mathcal{A}_{\mathrm{I}} + \frac{1}{2A}(2B\rho - \dot{\rho})\mathcal{A}_{\mathrm{R}}\right)q_p + \rho\mathcal{A}_{\mathrm{R}}p_p\right]. \tag{42}$$

In terms of $\hat{U}(q,p)$, the Wigner distribution function of density operator Eq. (10) is (Chountasis and Vourdas, 1998)

$$W(q,p,t) = \frac{1}{\pi\hbar}\mathrm{Tr}[\hat{\varrho}\hat{U}(q,p)] = \frac{1}{\pi\hbar}\sum_{n,m}\varrho_{nm}\langle\hat{U}(q,p)\rangle_{mn}, \tag{43}$$

where

$$\begin{aligned}\langle\hat{U}(q,p)\rangle_{mn} &= \langle\psi_m|\hat{U}(q,p)|\psi_n\rangle \\ &= (-1)^n\left(\frac{n!}{m!}\right)^{1/2} e^{-2I(q,p,t)/(\hbar\Omega)}L_n^{m-n}\left(\frac{4}{\hbar\Omega}I(q,p,t)\right) \\ &\quad\times\left\{\frac{2}{\sqrt{\hbar\Omega}}\left[\left(\frac{\Omega}{2\rho} + i\frac{1}{2A}(2B\rho - \dot{\rho})\right)(q-q_p) + i\rho(p-p_p)\right]\right\}^{m-n}.\end{aligned} \tag{44}$$

In terms of Eqs. (26) and (27), Eq. (44) can be represented in a simple form

$$\langle\hat{U}(|\mathcal{A}|,\Theta)\rangle_{mn} = (-1)^n\left(\frac{n!}{m!}\right)^{1/2}(2|\mathcal{A}|)^{m-n}e^{i(m-n)\Theta}e^{-2|\mathcal{A}|^2}L_n^{m-n}(4|\mathcal{A}|^2). \tag{45}$$

The Wigner distribution function may be used to evaluate the expectation value of a quantum operator $\hat{f}$ in arbitrary state ψ (Lee, 1995):

$$\langle\psi|\hat{f}|\psi\rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp W(q, p, t) f(q, p, t). \tag{46}$$

For example, expectation value of the Hamiltonian and invariant operator in Fock state are

$$\begin{aligned}
\langle\psi_n|\hat{H}|\psi_n\rangle &= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp W_n(q, p, t) H(q, p, t) \\
&= A\left[\frac{\hbar\sqrt{k}}{4\rho^2}(1 + Z^2)(2n + 1) + p_p^2\right] \\
&\quad + B[-\hbar Z(2n + 1) + 2q_p p_p] + C\left[\frac{\rho^2\hbar}{\sqrt{k}}(2n + 1) + q_p^2\right] \\
&\quad + Dq_p + Ep_p + F, \qquad (47)\\
\langle\psi_n|\hat{I}|\psi_n\rangle &= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp W_n(q, p, t) I(q, p, t) \\
&= \hbar\Omega\left(n + \frac{1}{2}\right), \qquad (48)
\end{aligned}$$

where

$$Z = \frac{\rho}{A\Omega}(2B\rho - \dot{\rho}). \tag{49}$$

In the calculations of Eqs. (47) and (48) we used (Magnus *et al.*, 1966)

$$L_n(x^2 + y^2) = \frac{(-1)^n}{2^{2n}} \sum_{m=0}^{n} \frac{1}{m!(n - m)!} H_{2(n-m)}(x) H_{2m}(y). \tag{50}$$

Equations (47) and (48) coincide with the results of ref. (Choi, 2004a) that obtained using other method.

3. DISTRIBUTION OF COHERENT, SQUEEZED, AND THERMAL STATES VIA WIGNER DISTRIBUTION FUNCTION

The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{51}$$

Multiplying $\langle q|$ to both side of the above equation from left, we obtain the position representation of the coherent state (Choi, 2004a):

$$\langle q|\alpha\rangle = \sqrt[4]{\frac{\Omega}{2\rho^2\hbar\pi}} \exp\left\{ -\frac{1}{2\rho\hbar}\left[\frac{\Omega}{2\rho}(\langle q\rangle - q)^2 + \frac{i}{A}(2B\rho - \dot{\rho})\left(\frac{1}{2}q^2 - \langle q\rangle q\right)\right] + \frac{i}{\hbar}\langle p\rangle q + i\delta_{c,q}\right\}, \quad (52)$$

where $\delta_{c,q}$ is some phase and $\langle q\rangle$ and $\langle p\rangle$ are expectation values in coherent state

$$\langle q\rangle = \langle\alpha|\hat{q}|\alpha\rangle = \rho\sqrt{\frac{\hbar}{\Omega}}(\alpha + \alpha^*) + q_p, \quad (53)$$

$$\langle p\rangle = \langle\alpha|\hat{p}|\alpha\rangle = \frac{\sqrt{\hbar\Omega}}{2i\rho}[(1 - iZ)\alpha - (1 + iZ)\alpha^*] + p_p. \quad (54)$$

From the above two equations, we see that the eigenvalue α is given by

$$\alpha = \frac{1}{2\rho}\sqrt{\frac{\Omega}{\hbar}}(1 + iZ)(\langle q\rangle - q_p) + i\rho\sqrt{\frac{1}{\hbar\Omega}}(\langle p\rangle - p_p). \quad (55)$$

The Wigner distribution function for the coherent state representation is

$$W_c(q, p, t) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\langle\alpha|q + y\rangle\langle q - y|\alpha\rangle e^{2ipy/\hbar}dy. \quad (56)$$

By making use of Eq. (52), the above equation can be evaluated as

$$W_c(q, p, t) = \frac{1}{\pi\hbar}\exp\left\{-\frac{\Omega}{2\hbar\rho^2}(q - \langle q\rangle)^2 - \frac{2\rho^2}{\hbar\Omega}\left[(p - \langle p\rangle) + \frac{2B\rho - \dot{\rho}}{2A\rho}(q - \langle q\rangle)\right]^2\right\}. \quad (57)$$

In order to investigate the Wigner distribution function for the squeezed state representation, let's consider the operator

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger, \quad (58)$$

where

$$|\mu|^2 - |\nu|^2 = 1. \quad (59)$$

We can easily see that $\hat{b}$ obeys $[\hat{b}, \hat{b}^\dagger] = 1$. The squeezed state $|\beta\rangle$ is eigenstate of $\hat{b}$:

$$\hat{b}|\beta\rangle = \beta|\beta\rangle. \tag{60}$$

The coordinate representation of the squeezed state is obtained by multiplying both sides of the above equation from the left by $\langle q|$

$$\langle q|\beta\rangle = N_q \exp\left\{ -\frac{1}{\rho\hbar}\left[\frac{\Pi}{\mu-\nu}\left(\frac{1}{2}q^2 - q_p q\right) - i\rho p_p q\right] + \frac{\mu\alpha + \nu\alpha^*}{\rho(\mu-\nu)}\sqrt{\frac{\Omega}{\hbar}}q\right\}, \tag{61}$$

where

$$N_q = \left(\frac{\Omega}{2\rho^2\hbar\pi}\frac{1}{(\mu-\nu)(\mu^*-\nu^*)}\right)^{1/4} \times \exp\left[-\frac{\Omega}{4\rho^2\hbar}\frac{1}{(\mu-\nu)(\mu^*-\nu^*)}\left(q_p + 2\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha\right)^2 + i\delta_{\mathrm{s},q}\right], \tag{62}$$

$$\Pi = \frac{\Omega}{2\rho}(\mu+\nu) + \frac{i(2B\rho - \dot{\rho})}{2A}(\mu-\nu), \tag{63}$$

with some phase $\delta_{\mathrm{s},q}$. Similarly, the momentum representation of the squeezed state is easily derived to be

$$\langle p|\beta\rangle = N_p \exp\left\{\frac{q_p}{i\hbar}p - \frac{1}{\hbar\Pi}\left[\frac{\rho}{2}(\mu-\nu)(p^2 - 2p_p p) + i\left(\mu\alpha + \nu\alpha^*\right)\sqrt{\hbar\Omega}p\right]\right\}, \tag{64}$$

where

$$N_p = \left(\frac{\Omega}{2\hbar\Pi^2\pi}\right)^{1/4} \exp\left\{-\frac{\Omega}{4\hbar|\Pi|^2}\left[p_p + \frac{1}{\rho}\sqrt{\hbar\Omega} \times \left(\mathrm{Im}\,\alpha - \frac{\rho}{A\Omega}(2B\rho - \dot{\rho})\mathrm{Re}\,\alpha\right)\right]^2 + i\delta_{\mathrm{s},p}\right\}, \tag{65}$$

with another some phase $\delta_{\mathrm{s},p}$. A simple phase space distribution of the squeezed state is

$$P_{\mathrm{s}}(q,p,t) = |\langle q|\beta\rangle|^2|\langle p|\beta\rangle|^2$$
$$= \frac{\Omega}{2\hbar|\Pi|\pi\rho}\frac{1}{[(\mu-\nu)(\mu^*-\nu^*)]^{1/2}}$$

$$\times \exp\left\{ -\frac{\Omega}{2\rho^2\hbar}\frac{1}{(\mu-\nu)(\mu^*-\nu^*)}\left(q-q_p-2\rho\sqrt{\frac{\hbar}{\Omega}}\mathrm{Re}\,\alpha\right)^2 \right.$$
$$\left. -\frac{\Omega}{2\hbar|\Pi|^2}\left[p-p_p-\frac{1}{\rho}\sqrt{\hbar\Omega}\left(\mathrm{Im}\,\alpha-\frac{\rho}{A\Omega}(2B\rho-\dot{\rho})\mathrm{Re}\,\alpha\right)\right]^2\right\}. \tag{66}$$

The Wigner distribution function for the squeezed state representation is

$$W_{\mathrm{s}}(q,p,t)=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\langle\beta|q+y\rangle\langle q-y|\beta\rangle e^{2ipy/\hbar}\,dy. \tag{67}$$

Substitution of Eq. (61) into the above equation gives

$$W_{\mathrm{s}}(q,p,t)=\frac{1}{\pi\hbar}\exp\left\{-\frac{\Omega}{2\hbar\rho^2}\frac{(q-\langle q\rangle)^2}{(\mu-\nu)(\mu^*-\nu^*)}-\frac{2\rho^2}{\hbar\Omega}(\mu-\nu)(\mu^*-\nu^*)\right.$$
$$\left.\times\left[(p-\langle p\rangle)+\left(\frac{2B\rho-\dot{\rho}}{2A\rho}-\frac{i\Omega}{2\rho^2}\frac{\mu^*\nu-\mu\nu^*}{(\mu-\nu)(\mu^*-\nu^*)}\right)(q-\langle q\rangle)\right]^2\right\}. \tag{68}$$

For $\mu=1$ and $\nu=0$, the above equation recovers to that of coherent state Eq. (57).

The probability $P_{n,\mathrm{s}}$ of finding n quanta in the squeezed state is given by

$$P_{n,\mathrm{s}}=2\pi\hbar\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}dp W_n(q,p,t)P_{\mathrm{s}}(q,p,t). \tag{69}$$

Now we introduce parameter c as $c=\mu/\nu$ with $|c|\geq 1$. In case that c is real, we readily evaluate the above equation using Eqs. (14) and (66) to be

$$P_{n,\mathrm{s}}=\frac{\sqrt{c^2-1}}{|c|^{n+1}}\frac{1}{2^n n!}H_n(y)H_n(y^*)$$
$$\times\exp\left\{-\frac{2}{c-1}\left[(c+1)(\mathrm{Re}\,\alpha)^2-\frac{c^2+1}{4c}(\alpha^2+\alpha^{*2})-|\alpha|^2\right]\right\}, \tag{70}$$

where

$$y=\frac{c\alpha+\alpha^*}{(2c)^{1/2}}. \tag{71}$$

In the calculation of Eq. (70) we used the integral formula Eq. (50). If we represent α as $\alpha = |\alpha| e^{i\varphi(t)}$, Eq. (70) becomes

$$P_{n,\mathrm{s}} = \frac{\sqrt{c^2-1}}{|c|^{n+1}} \frac{1}{2^n n!} H_n\left(\frac{|\alpha|}{(2c)^{1/2}}\left(ce^{i\varphi(t)} + e^{-i\varphi(t)}\right)\right) \times H_n\left(\frac{|\alpha|}{(2c)^{1/2}}\left(ce^{-i\varphi(t)} + e^{i\varphi(t)}\right)\right) \exp\left[-|\alpha|^2\left(1+\frac{1}{c}\cos[2\varphi(t)]\right)\right]. \tag{72}$$

This is same as that of Eq. (3.3) in ref. (Schleich and Wheeler, 1987).

Now we see the Wigner distribution function for the thermal state representation. The density operator that satisfying the Liouville–von Neumann equation in thermal state is given by (Ji and Kim, 1996; Choi, 2003)

$$\varrho_{\mathrm{T}}(q, q', t) = \frac{1}{Z(t)} \sum_{n=0}^{\infty} \psi_n(q,t) \exp\left[-\frac{\hbar\Omega_0}{kT}\left(n+\frac{1}{2}\right)\right] \psi_n^*(q', t), \tag{73}$$

where k is Boltzmann's constant, T the temperature of the system at initial time, $\Omega_0 = A(0)\Omega/\rho^2(0)$, and $Z(t)$ the partition function:

$$Z(t) = \sum_{n=0}^{\infty} \langle \psi_n(t)| e^{-\hbar\Omega_0(\hat{a}^\dagger \hat{a}+1/2)/(kT)} |\psi_n(t)\rangle. \tag{74}$$

Using Eq. (6), Eq. (73) can be readily calculated as

$$\varrho_{\mathrm{T}}(q, q', t) = \sqrt{\frac{\Omega}{2\rho^2\hbar\pi}}\left[\tanh\left(\frac{\hbar\Omega_0}{2kT}\right)\right]^{1/2} \exp\left[\frac{i}{\hbar} p_p (q-q')\right] \times \exp\left\{-i\frac{2B\rho-\dot{\rho}}{4\rho A\hbar}[(q-q_p)^2 - (q'-q_p)^2] - \frac{\Omega}{8\rho^2\hbar}\left[\tanh\left(\frac{\hbar\Omega_0}{2kT}\right)(q+q'-2q_p)^2 + \coth\left(\frac{\hbar\Omega_0}{2kT}\right)(q-q')^2\right]\right\}. \tag{75}$$

Then, we also easily derive the corresponding Wigner distribution function

$$W_{\mathrm{T}}(q, p, t) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \varrho_{\mathrm{T}}(q-y, q+y, t) e^{2ipy/\hbar}\, dy = \frac{1}{\pi\hbar}\tanh\left(\frac{\hbar\Omega_0}{2kT}\right) \exp\left\{-\tanh\left(\frac{\hbar\Omega_0}{2kT}\right)\left\{\frac{\Omega}{2\hbar\rho^2}(q-q_p)^2 + \frac{2\rho^2}{\hbar\Omega}\left[(p-p_p) + \frac{2B\rho-\dot{\rho}}{2A\rho}(q-q_p)\right]^2\right\}\right\}. \tag{76}$$

4. APPLICATIONS

Our theory may be applied to various kind of time-dependent Hamiltonian systems. As an example, let us see for the one-dimensional motion of Brownian particle (Guz *et al.*, 2003).

$$m\frac{d^2q}{dt^2} + \kappa\frac{dq}{dt} = \mathcal{F} - \mathcal{G}\sin\left[\frac{q}{l} + \xi(t)\right], \tag{77}$$

where q is coordinate of the particle whose mass is m, κ is a friction constant, $\mathcal{F}$ is a constant external force, $\mathcal{G}$ is an amplitude, $2\pi l$ is a spatial period, and $\xi(t)$ is a stationary random process describing phase fluctuations. To simplify the problem, we only consider the region that $q/l + \xi(t) \ll 1$. Then the above equation can be rewritten as

$$m\frac{d^2q}{dt^2} + \kappa\frac{dq}{dt} \simeq \mathcal{F} - \mathcal{G}\left[\frac{q}{l} + \xi(t)\right]. \tag{78}$$

We consider for the case of a stationary frequency modulation by supposing that the process $\dot{\xi}$ is an Ornstein–Uhlenbeck process so that

$$\ddot{\xi}(t) + \gamma\dot{\xi}(t) = \gamma\sqrt{\mathcal{D}}w, \tag{79}$$

where γ is an relaxation constant and $\mathcal{D}/2\pi$ is the spectral density of input Gaussian noise w. The general solution of the above equation is

$$\xi(t) = -c_1\frac{e^{-\gamma t}}{\gamma} + c_2 + \sqrt{\mathcal{D}}wt, \tag{80}$$

where c_1 and c_2 are integral constants. For this system, the Hamiltonian can expressed in the form

$$\hat{H} = e^{-\kappa t/m}\frac{\hat{p}^2}{2m} + e^{\kappa t/m}\frac{\mathcal{G}}{2l}\hat{q}^2 - me^{\kappa t/m}\left[\mathcal{F} + c_1\frac{e^{-\gamma t}}{\gamma} - c_2 - \sqrt{\mathcal{D}}wt\right]\hat{q}. \tag{81}$$

We can easily check that the above Hamiltonian gives coordinate equation of motion Eq. (78) using Hamiltonian dynamics. Then, Eq. (3) becomes

$$\ddot{\rho} + \frac{\kappa}{m}\dot{\rho} + \frac{\mathcal{G}}{ml}\rho - \frac{\Omega^2}{4m^2}e^{-2\kappa t/m}\frac{1}{\rho^3} = 0. \tag{82}$$

The solution of the above equation can be written as

$$\rho(t) = \frac{\tilde{E}^{\frac{1}{2}}(t)}{\omega_d\sqrt{m}}e^{-\kappa t/(2m)}, \tag{83}$$

where ω_d is given by

$$\omega_d = \sqrt{\frac{\mathcal{G}}{ml} - \frac{\kappa^2}{4m^2}}. \tag{84}$$

and

$$\tilde{E}(t) = \sqrt{E_0^2 - \frac{\Omega^2\omega_d^2}{4}\cos[2(\omega_d t + \theta)] + E_0} \tag{85}$$

with E_0 is an integral constant and θ is some phase. The particular solution q_p satisfy the following relations

$$m\ddot{q}_p + \kappa\dot{q}_p \simeq \mathcal{F} - \mathcal{G}\left(\frac{q_p}{l} - c_1\frac{e^{-\gamma t}}{\gamma} + c_2 + \sqrt{\mathcal{D}}wt\right), \tag{86}$$

which leads to

$$\begin{aligned} q_p(t) = {} & \frac{l}{\mathcal{G}\gamma[\mathcal{G} + \gamma l(m\gamma - \kappa)]}\big\{l\gamma^2(m\gamma - \kappa)(\mathcal{F} + \kappa l\sqrt{\mathcal{D}}w) \\ & + \mathcal{G}^2\big[c_1 e^{-\gamma t} - \gamma(c_2 + \sqrt{\mathcal{D}}wt)\big] + \mathcal{G}\gamma\big\{\mathcal{F} + l\big[c_2\gamma(\kappa - m\gamma) \\ & + \sqrt{\mathcal{D}}w(\kappa + \kappa\gamma t - m\gamma^2 t)\big]\big\}\big\}. \end{aligned} \tag{87}$$

From $\dot{q}_p = \partial\hat{H}/\partial p_p$, another particular solution in momentum space is given by

$$p_p(t) = me^{\kappa t/m}\frac{dq_p(t)}{dt}. \tag{88}$$

Then, Eqs. (57), (68) and (76) becomes

$$\begin{aligned} W_{\mathrm{c}}(q,p,t) = {} & \frac{1}{\pi\hbar}\exp\left\{-\frac{\Omega\omega_d^2 m}{2\hbar\tilde{E}(t)}e^{\kappa t/m}(q - \langle q\rangle)^2 - \frac{2\tilde{E}(t)}{\hbar\Omega\omega_d^2 m}\right. \\ & \times\left[(p - \langle p\rangle)e^{-\kappa t/(2m)} + e^{\kappa t/(2m)}\left(\frac{m\omega_d}{\tilde{E}(t)}\sqrt{E_0^2 - \frac{\Omega^2\omega_d^2}{4}}\right.\right. \\ & \left.\left.\left.\times\sin[2(\omega_d t + \theta)] + \frac{\kappa}{2}\right)(q - \langle q\rangle)\right]^2\right\}, \end{aligned} \tag{89}$$

$$\begin{aligned} W_{\mathrm{s}}(q,p,t) = {} & \frac{1}{\pi\hbar}\exp\left\{-\frac{\Omega\omega_d^2 m}{2\hbar\tilde{E}(t)}e^{\kappa t/m}\frac{(q - \langle q\rangle)^2}{(\mu - \nu)(\mu^* - \nu^*)}\right. \\ & -\frac{2\tilde{E}(t)}{\hbar\Omega\omega_d^2 m}(\mu - \nu)(\mu^* - \nu^*)\Big[(p - \langle p\rangle)e^{-\kappa t/(2m)} \end{aligned}$$

$$+ e^{\kappa t/(2m)}\left(\frac{m\omega_d}{\tilde{E}(t)}\sqrt{E_0^2 - \frac{\Omega^2\omega_d^2}{4}}\sin[2(\omega_d t + \theta)] + \frac{\kappa}{2}\right.$$

$$\left.\left. - \frac{i\Omega\omega_d^2 m}{2\tilde{E}(t)}\frac{\mu^*\nu - \mu\nu^*}{(\mu-\nu)(\mu^*-\nu^*)}\right)(q - \langle q\rangle)\Big]^2\right\}, \tag{90}$$

$$W_{\mathrm{T}}(q,p,t) = \frac{1}{\pi\hbar}\tanh\left(\frac{\hbar\Omega_0}{2kT}\right)\exp\left\{-\tanh\left(\frac{\hbar\Omega_0}{2kT}\right)\left\{\frac{\Omega\omega_d^2 m}{2\hbar\tilde{E}(t)}e^{\kappa t/m}(q-q_p)^2\right.\right.$$

$$+ \frac{2\tilde{E}(t)}{\hbar\Omega\omega_d^2 m}\left[(p-p_p)e^{-\kappa t/(2m)} + e^{\kappa t/(2m)}\left(\frac{m\omega_d}{\tilde{E}(t)}\sqrt{E_0^2 - \frac{\Omega^2\omega_d^2}{4}}\right.\right.$$

$$\left.\left.\left.\left.\times\sin[2(\omega_d t + \theta)] + \frac{\kappa}{2}\right)(q-q_p)\right]^2\right\}\right\}. \tag{91}$$

We presented quadrature plot of Wigner distribution function with the choice of $\Omega = 2E_0/\omega_d$ in Fock state (Fig. 1), in coherent state (Fig. 2), and in squeezed state (Fig. 3). The graph in Fig. 1 permits negative value as well as positive value while those in Figs. 2 and 3 are always positive. Figures 1(b), 2(b), and 3(b), shows the decrease of the position amplitude and the increase of momentum amplitude with time. These natural change of the amplitude of the oscillations for similar dissipative TDQHS has been reported in (Nieto and Truax, 2001).

If the dissipation and driving force disappears, i.e., $\kappa = 0$, $q_p = 0$, and $p_p = 0$ the system becomes standard harmonic oscillator. Then, with the choice of $\Omega = 2E_0/\omega_0$ where $\omega_0 = [\mathcal{G}/(ml)]^{1/2}$, Eqs. (89), (90), and (91) reduce to

$$W_{\mathrm{c}}(q,p,t) = \frac{1}{\pi\hbar}\exp\left[-\frac{m\omega_0}{\hbar}(q-\langle q\rangle)^2 - \frac{1}{\hbar m\omega_0}(p-\langle p\rangle)^2\right], \tag{92}$$

$$W_{\mathrm{s}}(q,p,t) = \frac{1}{\pi\hbar}\exp\left\{-\frac{m\omega_0}{\hbar}\frac{(q-\langle q\rangle)^2}{(\mu-\nu)(\mu^*-\nu^*)} - \frac{1}{\hbar m\omega_0}(\mu-\nu)(\mu^*-\nu^*)\right.$$

$$\left.\times\left[(p-\langle p\rangle) - im\omega_0\frac{\mu^*\nu-\mu\nu^*}{(\mu-\nu)(\mu^*-\nu^*)}(q-\langle q\rangle)\right]^2\right\}, \tag{93}$$

$$W_{\mathrm{T}}(q,p,t) = \frac{1}{\pi\hbar}\tanh\left(\frac{\hbar\omega_0}{2kT}\right)\exp\left\{-\tanh\left(\frac{\hbar\omega_0}{2kT}\right)\left[\frac{m\omega_0}{\hbar}q^2 + \frac{1}{\hbar m\omega_0}p^2\right]\right\}. \tag{94}$$

These are coincide to previous reports (Chountasis and Vourdas, 1998).

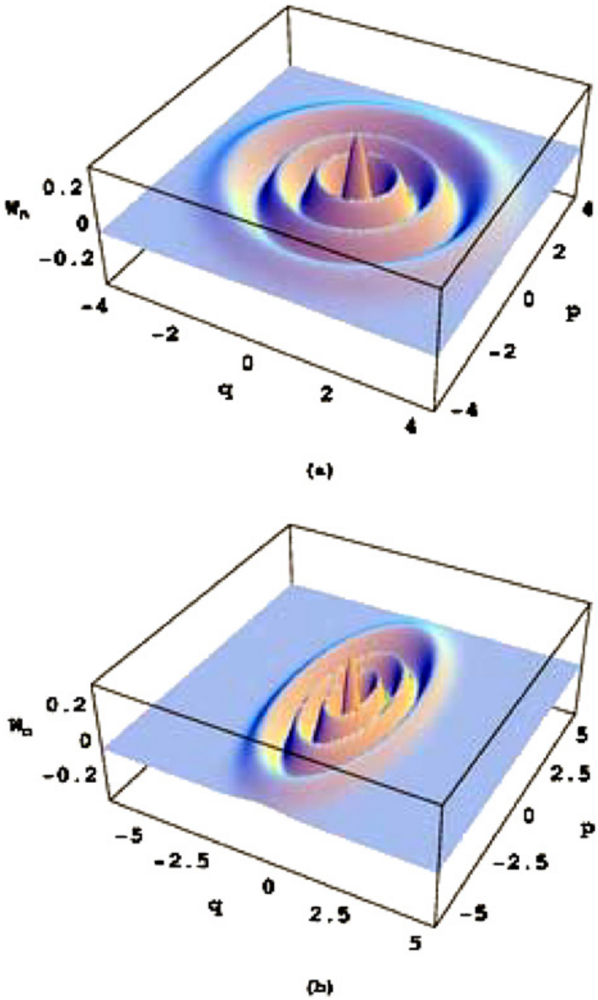

Fig. 1. Quadrature plot of Wigner distribution function in Fock state. The values of quantum number and time, (n, t), are (6,0) for (a), and (6,8) for (b). We used $c_1 = 0.1$, $c_2 = 0.1$, $w = 1$, $\kappa = 0.1$, $\gamma = 0.1$, $\mathcal{D} = 1$, $\mathcal{G} = 1$, $\mathcal{F} = 1$, $m = 1$, $\hbar = 1$, and $l = 1$.

Now we will apply our theory to the driven oscillator with strongly pulsating mass (Abdalla and Colegrave, 1985). In this case the Hamiltonian is given by

$$\hat{H}(\hat{q}, \hat{p}, t) = \frac{\hat{p}^2}{2M(t)} + \frac{1}{2}M(t)\left[\omega_0^2\hat{q}^2 - 2f(t)\hat{q}\right], \tag{95}$$

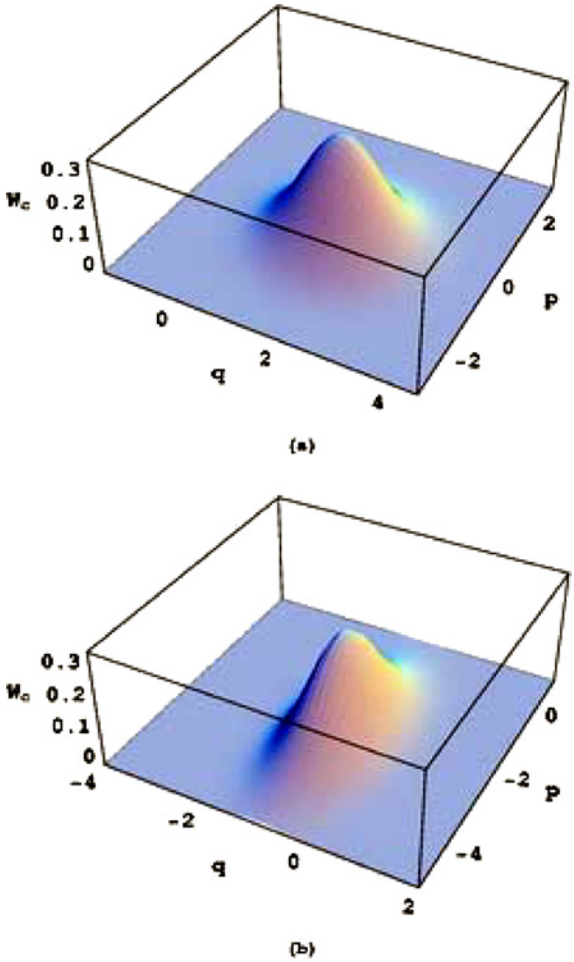

Fig. 2. Quadrature plot of Wigner distribution function in coherent state at $t = 0$(a) and $t = 8$(b). We supposed that $\alpha = |\alpha|e^{i\varphi}$ with $|\alpha| = 1$ and $\varphi = -\omega_d t$. We used $c_1 = 0.1$, $c_2 = 0.1$, $w = 1$, $\kappa = 0.1$ $\gamma = 0.1$, $\mathcal{D} = 1$, $\mathcal{G} = 1$, $\mathcal{F} = 1$, $m = 1$, $\hbar = 1$, and $l = 1$.

where

$$M(t) = m\cos^2 \omega_m t, \tag{96}$$

$$f(t) = f_0 \cos(\omega_f t + \vartheta), \tag{97}$$

with m is mass at $t = 0$, ω_m and ω_f are arbitrary constant frequencies, and f_0 and ϑ are amplitude and initial phase of the driving force. Then, Eq. (3)

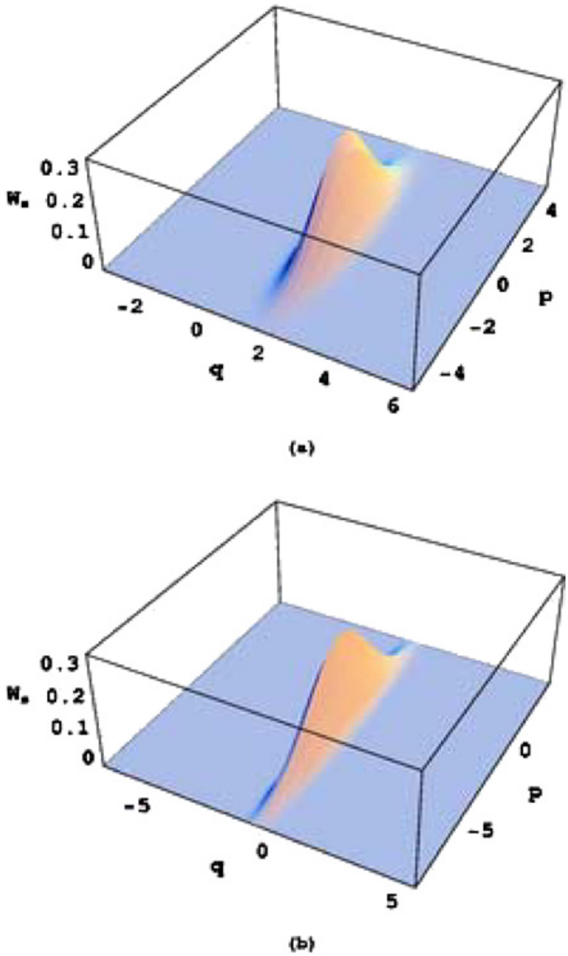

Fig. 3. Quadrature plot of Wigner distribution function in squeezed state. The values of (μ, ν, t) are (1.35, 0.9, 0) for (a) and (1.35, 0.9, 8) for (b). We imposed the same condition for α as in Fig. 2. We used $c_1 = 0.1$, $c_2 = 0.1$, $w = 1$, $\kappa = 0.1$, $\gamma = 0.1$, $\mathcal{D} = 1$, $\mathcal{G} = 1$, $\mathcal{F} = 1$, $m = 1$, $\hbar = 1$, and $l = 1$.

becomes

$$\ddot{\rho} - 2\omega_m \tan(\omega_m t)\dot{\rho} + \omega_0^2 \rho - \frac{\Omega^2}{4M^2(t)} \frac{1}{\rho^3} = 0. \tag{98}$$

The solution of the above equation is

$$\rho(t) = \frac{\tilde{\mathsf{E}}^{\frac{1}{2}}(t)}{\omega\sqrt{m}} \sec(\omega_m t), \tag{99}$$

where ω is given by

$$\omega = \sqrt{\omega_0^2 + \omega_m^2}, \tag{100}$$

and

$$\tilde{\mathsf{E}}(t) = \sqrt{E_0^2 - \frac{\Omega^2\omega^2}{4}} \cos[2(\omega t + \theta)] + E_0, \tag{101}$$

with E_0 is an integral constant and θ is some phase. On the other hand, the equations that particular solutions q_p and q_p should follow are

$$\ddot{q}_p - 2\omega_m \tan(\omega_m t)\dot{q}_p + \omega_0^2 q_p = f_0 \cos(\omega_f t + \vartheta), \tag{102}$$

$$\ddot{p}_p + 2\omega_m \tan(\omega_m t)\dot{p}_p + \omega_0^2 p_p = -m f_0 \omega_f \cos^2(\omega_m t) \sin(\omega_f t + \vartheta). \tag{103}$$

By solving the above two equations, the particular solutions become (Abdalla and Colegrave, 1985)

$$q_p(t) = \frac{1}{2} f_0 \sec(\omega_m t)\left(\frac{\cos[(\omega_f + \omega_m)t + \vartheta] - \cos(\omega t)\cos\vartheta}{\omega^2 - (\omega_f + \omega_m)^2} + \frac{\cos[(\omega_f - \omega_m)t + \vartheta] - \cos(\omega t)\cos\vartheta}{\omega^2 - (\omega_f - \omega_m)^2}\right), \tag{104}$$

$$p_p(t) = [mM(t)]^{1/2}\left(\frac{d}{dt}(q_p \cos\omega_m t) + \omega_m q_p \sin\omega_m t\right). \tag{105}$$

In terms of Eqs. (99), (104), and (105) the quantum solution of the system can be completely described and the Wigner distribution functions becomes

$$W_{\mathrm{c}}(q, p, t) = \frac{1}{\pi\hbar} \exp\left\{ -\frac{\Omega\omega^2 m}{2\hbar\tilde{\mathsf{E}}(t)} \cos^2(\omega_m t)(q - \langle q\rangle)^2 - \frac{2\tilde{\mathsf{E}}(t)}{\hbar\Omega\omega^2 m} \times\left[(p - \langle p\rangle)\sec(\omega_m t) + m\cos(\omega_m t)\left(\frac{\omega}{\tilde{\mathsf{E}}(t)}\sqrt{E_0^2 - \frac{\Omega^2\omega^2}{4}} \times \sin[2(\omega t + \theta)] - \omega_m \tan(\omega_m t)\right)(q - \langle q\rangle)\right]^2\right\}, \tag{106}$$

$$W_{\mathrm{s}}(q, p, t) = \frac{1}{\pi\hbar} \exp\left\{ -\frac{\Omega\omega^2 m}{2\hbar\tilde{\mathsf{E}}(t)} \frac{\cos^2(\omega_m t)}{(\mu - \nu)(\mu^* - \nu^*)}(q - \langle q\rangle)^2 - \frac{2\tilde{\mathsf{E}}(t)}{\hbar\Omega\omega^2 m}(\mu - \nu)(\mu^* - \nu^*)\left[(p - \langle p\rangle)\sec(\omega_m t)\right.\right.$$

$$+\cos(\omega_m t)\left(\frac{m\omega}{\tilde{\mathsf{E}}(t)}\sqrt{E_0^2-\frac{\Omega^2\omega^2}{4}}\sin[2(\omega t+\theta)]-m\omega_m\tan(\omega_m t)\right.$$

$$\left.\left.\left.-\frac{i\Omega\omega^2 m}{2\tilde{\mathsf{E}}(t)}\frac{\mu^*\nu-\mu\nu^*}{(\mu-\nu)(\mu^*-\nu^*)}\right)(q-\langle q\rangle)\right]^2\right\}, \tag{107}$$

$$W_{\mathrm{T}}(q,p,t)=\frac{1}{\pi\hbar}\tanh\left(\frac{\hbar\Omega_0}{2kT}\right)\exp\left\{-\tanh\left(\frac{\hbar\Omega_0}{2kT}\right)\left\{\frac{\Omega\omega^2 m}{2\hbar\tilde{\mathsf{E}}(t)}\cos^2(\omega_m t)(q-q_p)^2\right.\right.$$

$$+\frac{2\tilde{\mathsf{E}}(t)}{\hbar\Omega\omega^2 m}\left[(p-p_p)\sec(\omega_m t)+m\cos(\omega_m t)\left(\frac{\omega}{\tilde{\mathsf{E}}(t)}\sqrt{E_0^2-\frac{\Omega^2\omega^2}{4}}\right.\right.$$

$$\left.\left.\left.\left.\times\sin[2(\omega t+\theta)]-\omega_m\tan(\omega_m t)\right)(q-q_p)\right]^2\right\}\right\}. \tag{108}$$

5. SUMMARY

We derived Wigner distribution function of the general TDQHS whose Hamiltonian is given by Eq. (1) according to the exact principle of quantum mechanics. The Wigner distribution function for the Fock state, the coherent state, the squeezed state, and the thermal state are given by Eqs. (16), (57), (68), and (76), respectively. The Wigner distribution function for the coherent state, squeezed state, and thermal state is Gaussian while that of the Fock state is non-Gaussian and expressed in terms of the Laguerre polynomial. From figures we can see that the Wigner distribution function of the system whose Hamiltonian is explicitly depend on time follows the rule that the only pure states for which the Wigner distribution function is everywhere positive are those for which the wave function satisfying Schrödinger equation is Gaussian. Our development of the Wigner distribution function may be used to evaluate the expectation value of various quantum operators in arbitrary state. We evaluated the probability $P_{n,\mathrm{s}}$ of finding n quanta in squeezed state in Eq. (72) which agree with other reports (Schleich and Wheeler, 1987). Our investigations are applied to the one-dimensional motion of Brownian particle and to the driven oscillator with strongly pulsating mass. We leave the research of Wigner distribution function for the superposition of two coherent states as a later task. We will do it in the near future.

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